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ON A CONGRUENTIAL PROPERTY OF FIBONACCI NUMBERS — CONSIDERATIONS AND REMARKS —

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In a previous note with the same title [4], we have examined on a computer the possibility of the converse of the following proposition, obtaining several counter examples to it :

Proposition. *If N is a prime number $\neq 5$, then we have*

$$U_N \equiv \left(\frac{N}{5} \right) \pmod{N},$$

where U_N denotes the N -th Fibonacci number and $(N/5)$ is the Legendre symbol.

In the present note, we shall try to weaken the restriction for N to be a prime, thus extending this proposition. We have found, as a result, that this can actually be achieved to some extent (cf. our Theorem below).

We define the numbers V_n ($n = 1, 2, \dots$) by setting

$$V_n = \frac{U_{2n}}{U_n}.$$

The sequence (V_n) is known as the so-called associated Lucas sequence [1, 2]. The numbers U_n and V_n can be written in the form, with $a = (1 + \sqrt{5})/2$, $b = (1 - \sqrt{5})/2$,

$$(1) \quad U_n = \frac{a^n - b^n}{\sqrt{5}}, \quad V_n = a^n + b^n \quad (n = 1, 2, \dots).$$

If an integer $N > 0$, $\neq 5$, satisfies the relation

$$U_N \equiv \left(\frac{N}{5} \right) \pmod{N},$$

we shall call N a converse number. When a prime number p retains the property that some U_n with $n > 1$ is divisible by p , but for any integer m , $0 < m < n$, U_m is not divisible by p , we call p a primitive (prime) factor of U_n . It is known that every Fibonacci number U_n with $n \neq 1, 2, 6$ or 12 admits a primitive factor (cf. [3; §2]). A prime factor

q of U_n which is not primitive will be called an elementary factor of U_n .

The proper part of U_n is the number which is obtained by removing all the elementary factors from U_n . In the subsequent discussions, we always assume that $n > 5$ in U_n , so that the proper part of any U_n under consideration is an odd integer which does not contain the number 5 as a factor.

Lemma 1. *If p is a primitive factor of a number U_n , then p can be written in the linear form :*

$$\begin{aligned} p &= nk + 1 & \text{if } \left(\frac{p}{5}\right) &= 1, \\ p &= nk - 1 & \text{if } \left(\frac{p}{5}\right) &= -1 \end{aligned}$$

with some positive integer k .

This lemma is an immediate consequence of the proposition quoted in the previous note [4] and is a well-known result.

Lemma 2. *If N is a divisor of the proper part of some U_n , then N can be written in the form :*

$$\begin{aligned} N &= nk + 1 & \text{if } \left(\frac{N}{5}\right) &= 1, \\ N &= nk - 1 & \text{if } \left(\frac{N}{5}\right) &= -1 \end{aligned}$$

with some positive integer k .

Proof. We shall prove the lemma in the case where N is the product of two primitive factors p_1 and p_2 of U_n ; the general case can be treated in quite a similar way. Put $s_1 = (p_1/5)$, $s_2 = (p_2/5)$ and $s = (N/5) = s_1 s_2$. Since, by virtue of Lemma 1, $p_1 = nk_1 + s_1$ and $p_2 = nk_2 + s_2$ for some $k_1, k_2 > 0$, we have then

$$(2) \quad N = p_1 p_2 = n(nk_1 k_2 + s_2 k_1 + s_1 k_2) + s_1 s_2.$$

From this expression, the assertion is obvious.

Lemma 3. *For any odd integer $N > 0$, there hold the following relations :*

$$(3) \quad U_N - (-1)^{(N-1)/2} = U_{(N-1)/2} V_{(N+1)/2},$$

$$(4) \quad U_N - (-1)^{(N+1)/2} = U_{(N+1)/2} V_{(N-1)/2}.$$

Proof. By the expressions in (1), we have for any integer $n > 0$

$$\begin{aligned} U_n V_{n+1} &= \frac{a^n - b^n}{\sqrt{5}} (a^{n+1} - b^{n+1}) \\ &= \frac{a^{2n+1} - b^{2n+1}}{\sqrt{5}} - (ab)^n \frac{a-b}{\sqrt{5}} \\ &= U_{2n+1} - (-1)^n, \end{aligned}$$

giving the first relation in the lemma. The proof for the second relation is quite the same.

Lemma 4. *If k is an odd integer > 0 , then V_{kn} is divisible by V_n .*

Proof. By the second expression in (1) we have $V_n = a^n + b^n$ and $V_{kn} = a^{kn} + b^{kn}$, so that

$$\begin{aligned} \frac{V_{kn}}{V_n} &= a^{(k-1)n} + b^{(k-1)n} - (ab)^n (a^{(k-3)n} + b^{(k-3)n}) + \dots \\ &\quad + (-1)^{(k-1)/2} (ab)^{(k-1)n/2}, \end{aligned}$$

whence the result.

In the subsequent lemmas, N denotes a divisor of the proper part of some U_n , $n > 5$, and we put $s = (N/5)$.

Lemma 5. *$U_{(N-s)/2}$ is divisible by N if $(N-s)/n$ is even, and $V_{(N-s)/2}$ is divisible by N if $(N-s)/n$ is odd.*

Proof. Note that $(N-s)/n$ is integral, by Lemma 2. We examine four cases according to the sign of s and the parity of $k = (N-s)/n$.

1) The case of $s = 1$ and k even. In this case, $(N-1)/2$ is divisible by n . Therefore, $U_{(N-1)/2}$ is divisible by U_n . Consequently, $U_{(N-1)/2}$ is divisible by N .

2) The case of $s = 1$ and k odd. This case is possible only when n is even, and $(N-1)/2$ is an odd multiple of $n/2$. Therefore, $V_{(N-1)/2}$ is divisible by $V_{n/2}$, by Lemma 4. Besides, when n is even, all primitive factors of U_n are always factors of $V_{n/2}$. Hence, $V_{(N-1)/2}$ is divisible by N .

3) The case of $s = -1$ and k even. In this case, $(N+1)/2$ is divisible by n . Therefore, $U_{(N+1)/2}$ is divisible by U_n . Hence, $U_{(N+1)/2}$ is divisible by N .

4) The case of $s = -1$ and k odd. This case is possible only when n is even. Then, as in the case 2), $V_{(N+1)/2}$ is divisible by $V_{n/2}$, and hence $V_{(N+1)/2}$ is divisible by N .

Lemma 6. *If n is odd, then N has the linear form :*

$$\begin{aligned} N &= 4nk + 1 && \text{if } s = 1, \\ N &= 4nk + 2n - 1 && \text{if } s = -1. \end{aligned}$$

Hence, $(N-s)/n$ is always even.

Proof. We have $U_{2m+1} = U_m^2 + U_{m+1}^2$ for any integer $m > 0$. Therefore, N is a divisor of an integer of the form $x^2 + y^2$, where $(x, y) = 1$. Hence, in view of the property of quadratic residues, we must have $N \equiv 1 \pmod{4}$. From this fact the lemma will follow easily.

Lemma 7. *When $n \equiv 0 \pmod{4}$, then $(N-1)/n$ is even if $s = 1$, and $(N+1)/2$ is odd if $s = -1$.*

When $n \equiv 2 \pmod{4}$, then we always have $s = 1$, and $(N-1)/n$ may either be even or be odd.

Proof. Firstly, we examine the case where N consists only of one primitive factor p of U_n . And we classify the case into three.

1) The case of $n \equiv 0 \pmod{4}$ and $s = 1$. Assume that $(p-1)/n$ is odd. By the assumption and by Lemma 5, $V_{(p-1)/2}$ is divisible by p and $(p-1)/2$ is even. We have, by (4),

$$U_p - (-1)^{(p+1)/2} = U_{(p+1)/2} V_{(p-1)/2},$$

where $U_p \equiv 1 \pmod{p}$ since p is a prime. Therefore, in the above expression,

$$\begin{aligned} \text{the left-hand side} &\equiv 2 \pmod{p}, \text{ and} \\ \text{the right-hand side} &\equiv 0 \pmod{p}. \end{aligned}$$

This is impossible. Hence, $(p-1)/n$ must be even.

2) The case of $n \equiv 0 \pmod{4}$ and $s = -1$. Assume that $(p+1)/n$ is even. Then $U_{(p-1)/2}$ is divisible by p and $(p+1)/2$ is even. From (4) we see that

$$U_p - (-1)^{(p+1)/2} = U_{(p+1)/2} V_{(p-1)/2},$$

where

$$\begin{aligned} \text{the left-hand side} &\equiv -2 \pmod{p}, \text{ and} \\ \text{the right-hand side} &\equiv 0 \pmod{p}. \end{aligned}$$

The contradiction assures that $(p+1)/n$ is odd.

3) The case of $n \equiv 2 \pmod{4}$. Assume that $s = -1$. If we suppose in addition that $(p+1)/n$ is even, then, in like manner as in the case 2), we arrive at a contradiction. In the sequel, we shall suppose that $(p+1)/n$ is odd. Then, $(p+1)/2$ is odd and $V_{(p+1)/2}$ is divisible by p .

From (3)

$$U_p - (-1)^{(p-1)/2} = U_{(p-1)/2} V_{(p+1)/2};$$

here

the left-hand side $\equiv -2 \pmod{p}$, and

the right-hand side $\equiv 0 \pmod{p}$.

Hence, it must be that $s=1$. About the fact that $(N-1)/n$ may be even or odd, we can readily confirm it by examples.

Now, for the proof of the general case it will suffice only to consider the case where N consists of two primitive factors p_1 and p_2 of U_n . Put $s_1 = (p_1/5)$, $s_2 = (p_2/5)$ and $p_1 = nk_1 + s_1$, $p_2 = nk_2 + s_2$.

4) The case of $n \equiv 0 \pmod{4}$. By the expression (2), we have

$$\frac{N-s}{n} = nk_1k_2 + s_2k_1 + s_1k_2.$$

Since the first term nk_1k_2 is even, as can be verified by arguing like above, the following scheme of implications clarifies all of the case:

$s = 1$:

$s_1 = s_2 = 1 \Rightarrow k_1, k_2 \text{ even} \Rightarrow k_1 + k_2 \text{ even};$

$s_1 = s_2 = -1 \Rightarrow k_1, k_2 \text{ odd} \Rightarrow k_1 + k_2 \text{ even};$

$s = -1$:

$s_1 = 1, s_2 = -1 \Rightarrow k_1 \text{ even}, k_2 \text{ odd} \Rightarrow k_1 - k_2 \text{ odd};$

$s_1 = -1, s_2 = 1 \Rightarrow k_1 \text{ odd}, k_2 \text{ even} \Rightarrow -k_1 + k_2 \text{ odd}.$

5) The case of $n \equiv 2 \pmod{4}$.

From 3) we see that it is always true that $s_1 = s_2 = 1$, and we obtain $s = s_1s_2 = 1$.

Now, our main result can be formulated in the following

Theorem. *If N is a divisor of the proper part of a Fibonacci number U_n with $n > 5$, then N is a converse number.*

Proof. We divide the proof into five cases according to the sign of $s = (N/5)$ and the parity of n .

1) The case of n odd and $s = 1$. In this case, by Lemma 6, $(N-s)/n$ is always even and, by Lemma 5, $U_{(N-1)/2}$ is divisible by N . In addition, we have $N = 4nk + 1$ by Lemma 6, so that $(N-1)/2$ is even. From the formula (3) we see

$$U_N - (-1)^{(N-1)/2} = U_{(N-1)/2} V_{(N+1)/2} \equiv 0 \pmod{N}.$$

Hence, we have $U_N \equiv 1 \pmod{N}$.

2) The case of n odd and $s = -1$. In this case, N has the linear form $N = 4nk + 2n - 1$ by Lemma 6, so that $(N+1)/n$ is even. And,

$U_{(N+1)/2}$ is divisible by N and $(N+1)/2$ is odd. We have by (4)

$$U_N - (-1)^{(N+1)/2} = U_{(N+1)/2} V_{(N+1)/2} \equiv 0 \pmod{N}.$$

Hence, $U_N \equiv -1 \pmod{N}$.

3) The case of $n \equiv 0 \pmod{4}$ and $s = 1$. In this case, $(N-1)/n$ is even by Lemma 7. Therefore, $U_{(N+1)/2}$ is divisible by N and $(N-1)/2$ is even. We have by (3)

$$U_N - (-1)^{(N-1)/2} = U_{(N-1)/2} V_{(N+1)/2} \equiv 0 \pmod{N}.$$

Hence, $U_N \equiv 1 \pmod{N}$.

4) The case of $n \equiv 0 \pmod{4}$ and $s = -1$. In this case, $(N+1)/n$ is odd by Lemma 7. Therefore, $V_{(N+1)/2}$ is divisible by N and $(N+1)/2$ is even. By (3)

$$U_N - (-1)^{(N-1)/2} = U_{(N-1)/2} V_{(N+1)/2} \equiv 0 \pmod{N}.$$

Hence, $U_N \equiv -1 \pmod{N}$.

5) The case of $n \equiv 2 \pmod{4}$. In this case, we always have $s = 1$. If $(N-1)/n$ is even, then $U_{(N-1)/2}$ is divisible by N and $(N-1)/2$ is even. By (3)

$$U_N - (-1)^{(N-1)/2} = U_{(N-1)/2} V_{(N+1)/2} \equiv 0 \pmod{N}.$$

Hence, $U_N \equiv 1 \pmod{N}$.

On the other hand, if $(N-1)/n$ is odd, then $V_{(N-1)/2}$ is divisible by N and $(N-1)/2$ is odd. We have by (4)

$$U_N - (-1)^{(N+1)/2} = U_{(N+1)/2} V_{(N-1)/2} \equiv 0 \pmod{N}.$$

Hence, $U_N \equiv 1 \pmod{N}$.

The proof of our theorem is now complete.

Numerical example 1. $N = 4181 = 37 \cdot 113$ is the proper part of U_{19} . Hence, N is a composite converse number. This example, which gives the least composite converse number, is one of the examples we have listed in the previous report [4].

Numerical example 2. $N = 192900153617 = 2269 \cdot 4373 \cdot 19441$ is the proper part of U_{81} . Hence, $N_1 = 2269 \cdot 4373$, $N_2 = 2269 \cdot 19441$ and $N_3 = 4373 \cdot 19441$, together with N , are all composite converse numbers.

Remark. The contraposition of the classical proposition cited in the first paragraph of this note gives rise to the following

Criterion. *If an integer $N \not\equiv 0 \pmod{5}$ satisfies the relation*

$$U_N \not\equiv \left(\frac{N}{5}\right) \pmod{N},$$

then N is a composite number.

This criterion seems to be effective for the problem of factoring a large integer. However, our theorem shows that the above criterion does not provide any information, in the case of factorization of Fibonacci numbers at least.

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